The Reliability of Surface Fitting Methods in Orthometric Height Determination from GPS Observations

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Key words: Interpolation, Conventional outlier tests, Robust Estimation

SUMMARY

Nowadays, satellite based Global Positioning Systems (GPS) which attain gradually widespread usage field, provides important contributions in a surveying or engineering applications. Three-dimensional coordinates of points are obtained accurately and quickly with GPS observations, in the geosentric World Geodetic System 1984 (WGS84). However, height measured with GPS is ellipsoidal height. Ellipsoidal heights which are defined in geometrical sense, don't have any physical meaning, must be transformed to orthometric heights because of having been used in practical surveying. For this, it is necessary that geoid undulation is known. In this study, It is used interpolation with polynomial methods in deriving orthometric height from GPS/levelling in a study area. The reliability of the data is also investigated in terms of conventional outlier test and robust statistics and It is aimed that the most appropriate surface models are determined for chosen study area.

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1. INTRODUCTION

Geometric levelling has been the dominant technique for precise determination of the heights of points on the earth surface. It is a simple but labor-intensive and time consuming process. In contrast to geometric levelling, the GPS survey is an all-weather technique that is rapid and easy to operate (Zhan-ji and Yong-qi, 1999). The disadvantage of GPS surveying is that GPS derived heights are ellipsoidal heights with respect to its reference ellipsoid, e.g WGS 84. As ellipsoidal heights aren't used in a practical surveying, they must be transformed orthometric heights.

To determine an orthometric height, it is necessary that geoid undulation is known. There are two principal approaches with which to transform GPS ellipsoidal heights to orthometric heights. These comprise a gravimetrically determined geoid model, and interpolation between geometrically derived geoid heights where GPS measurements have been co-located with benchmarks (Featherstone et al., 1998).

There are many interpolation methods to transform ellipsoidal heights to orthometric heights. In this study, Cubic and quadratic interpolation methods are used a local study area, which is 46*51 km^2. The study area consists of 77 points whose points are known in WGS84 ellipsoid and Turkish National Datum. Forty four of them are selected reference points and remaining thirty three points are selected test points. By recalculating geoid undulations at the 33 test points with interpolation methods, interpolated values are compared with known values. In conclusion, these two interpolation methods are evaluated.

2. HEIGHT RELATIONSHIPS

The orthometric height of a point is the distance from the geoid or a related reference surface to the point on the earth's surface, measured along the direction of the plumb line perpendicular to every equipotential surface. The ellipsoidal height of a point is the distance of the earth's surface above or below the ellipsoid, measured along the normal to the ellipsoid. Ellipsoidal height is geometric meaning whereas orthometric height has a physical meaning and depends on the gravity field of the earth.

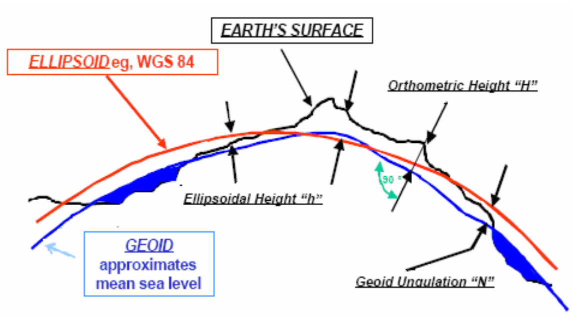

Figure 1. Relationship between ellipsoidal and orthometric heights (http://www.usace.army.mil/usace-docs/eng-manuals/em1110-1-1003/c-3.pdf, accessed February 2006)

The distance between the geoid and the ellipsoid is defined geoid undulation or geoid height. As can be seen from Fig.1, the geoid undulation , simply given by:

$$N = h - H \tag{1}$$

Where,
N = geoid undulation
h = ellipsoidal height
H = orthometric height

3. POLYNOMIAL INTERPOLATION

This method is the most widely used surface-fitting procedure. To model a surface, a mathematical function needs to be used. The polynomial can be expanded to any desired degree. The general mathematical expression of nth-degree polynomial is as follows (İnal, 1996):

$$N(x, y) = \sum_{i=1}^{n} \sum_{\substack{j=k-i \\ i=0}}^{k} a_{ij}\, x^i y^j \tag{2}$$

Where,
(x, y) = coordinates of an interpolation point,
a_{ij} = degree of polynomial,
n = degree of the chosen polynomial.

In the equation (2), when the degree of the polynomial is equal to 1, 2 and 3 then the fitted surface is called linear, quadratic and cubic respectively. In this study, second and third degree polynomial trend surfaces are used. For the quadratic and cubic trend surfaces, the general equations are as follow.

$$z(x, y) = a_{00} + a_{01}y + a_{10}x + a_{20}x^2 + a_{11}xy + a_{02}y^2 \tag{3}$$

$$z(x,y) = a_{00} + a_{01}y + a_{10}x + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 \quad (4)$$

In order to determine quadratic and cubic surface equations, the minimum number of the reference points is 6 and 9 points respectively. For redundant reference points, the unknown polynomial coefficients can be determined by the least-squares method according to

$$\underline{X} = (\underline{A}^T \underline{A})^{-1} \underline{A}^T \underline{L} \quad (5)$$

Where,
A = coefficient matrix,
L = observation vector, showing geoid undulation values belong to reference points.

4. DETECTION OF OUTLIERS

4.1. Conventional Outlier Detection Methods

The basis idea of conventional outlier detection tests is based on principle of being only one outlier in data set. If there are more one outliers among data set, process of outlier determination is done iteratively by starting from measurement having the largest residual in the data set and carried out until no more outliers are determined by eliminating single outlier in data set for each iteration step. To fulfill such a process according to mathematic statistic laws (Konak, 1994);

$$\begin{aligned} H_0 &: E\{v\} = 0 \quad \textit{Null Hypothesis} \\ H_s &: E\{v\} \neq 0 \quad \textit{Alternative Hypothesis} \end{aligned} \quad (6)$$

is utilized from a hypothesis established as above. Normalized random variable values for normalized distribution, namely test sizes are computed as follows:

Table 1. Test sizes and distributions

Method	Test size		Critical value
W-Test (Data-Snooping)	$W_i = \frac{\lvert v_i \rvert}{\sigma_{v_i}}$	$W_i = \frac{\lvert v_i \rvert}{\sigma_0 \sqrt{Q_{v_i v_i}}}$	$N_{1-\alpha/2} = \sqrt{F_{1,\infty,1-\alpha}}$
Tau Test (Poppe)	$T_i = \frac{\lvert v_i \rvert}{m_{v_i}}$	$T_i = \frac{\lvert v_i \rvert}{m_0 \sqrt{Q_{v_i v_i}}}$	$\tau_{f,1-\alpha/2} = \sqrt{\frac{f t^2_{f-1,1-\alpha/2}}{f-1+t^2_{f-1,1-\alpha/2}}}$
t-Test	$t_i = \frac{\lvert v_i \rvert}{\overline{m}_{v_i}}$	$t_i = \frac{\lvert v_i \rvert}{\overline{m}_{01} \sqrt{Q_{v_i v_i}}}$	$t_{f-1,1-\alpha/2} = \sqrt{F_{1,f-1,1-\alpha}}$

Where test values are analyzed by comparing them with critical value from distribution tables as to degree of freedom *f=n-u* and chosen confidence interval *α =1-S* in advance. If null hypothesis is rejected, observations are said to be outlier. Conversely, observations are said to be consistent.

4.2. Robust Estimation Method

One outlier is enough to ruin all results of Least Squares Method. Thus, conventional test methods based on least squares method cannot detect two or more outliers. To bound the influence of outliers on all results, Huber (1964) introduced the robust M-estimator which is generalized form of maximum likelihood and which is another approach used in parameter estimation. In this method, score function; $M = \sum_{i=1}^{n} \rho(v_i) = \min$.is selected and hence with derivative of $A^T \psi(v) = 0$ function with respect to unknown parameters, it is obtained weight function $W = W(v) = \psi(v)/v$**.** By utilizing this function**,** similir to Least Squares Method, the normal equations $A^T Wv = A^T W(Ax - l) = 0$ are found. The vector of unknowns from normal equations is obtained as follows (Hekimoğlu, 1999).

$$x = (A^T WA)^{-1} A^T Wl \qquad (7)$$

Because the residuals in *W(v)* are not known, this equation cannot be solved directly. However, it can be solved easily if the following iterative reweighted least square method is applied by choosing a proper function for $\rho(v)$. The computation is started with $W(v_0) = I$ and at each step, the new weights are computed iteratively. Such an approach consists of basis of robust weighted solution. Robust solutions have displayed different approaches in that different functions for weight parameter are chosen in each robust solution. In this study; Danish method proposed by Krarup has been chosen. This weight function is given as follows;

$$W(v_i) = \begin{bmatrix} \left(e^{-\frac{|v|}{c}} \right) & ; for\ |v| > c \\ 1 & ; for\ |v| \le c \end{bmatrix} \qquad (8)$$

For the computations, by utilizing t-student distribution $\frac{|v_i|}{m_o \sqrt{Q_{v_i v_i}}} \le t_{f,1-\alpha/2}$ and by choosing $c = |v_i|$, the constant c is taken as (Dilaver, Konak, Çepni, 1998).

$$c = m_0 \sqrt{Q_{v_i v_i}}\ t_{f,1-\frac{\alpha}{2}} \qquad (9)$$

5. NUMERICAL APPLICATION

The data used in this study are obtained from MERLIS project. The study area is about 46*51 km^2. It consists of seventy seven points whose points are known ellipsoid and orthometric heights, forty four of them are selected reference points and remaining thirty three points are selected test points. The distributions of the points can bee seen in Fig 2. The test points are determined by make use of digital elevation model of the study area as to topographic features.

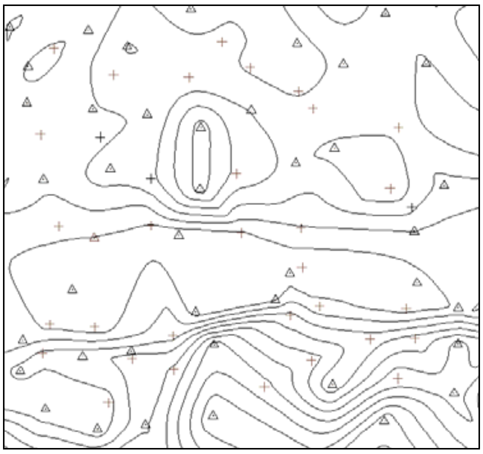

Figure 2. The distributions of the reference points and test points in the study area

It is used cubic and quadratic polynomials for polynomial surfaces in this study. According to 44 reference points, parameter tests and outlier tests are made. To detect outlier, $\tau - Tau\ \ test$ (by choosing significance level $\alpha = 0,05$) out of conventional methods, robust estimation approach have been applied to reference points separately for two methods. Consequently, in the result of conventional test, while three points are determined as outliers for the cubic trend surface, there isn't any outlier for quadratic trend surface. Outliers determined in robust estimation are similar to the ones determined in conventional test (Table 2).

Table 2. Outlier Results Determined in Cubic Trend Surface

Cubic Interpolation Method					
	Outlier Testing(Tau Test)			Robust Weights (*Wi*)	
P. N.	**ite no**	T_i	τ	**ite 1**	**ite 4**
25	1	3.44	1.95	0.18	0.00
42	2	2.80	1.95	0.35	0.05
24	3	2.47	1.95		

In addition, after outlier observations are determined, null hypothesis test results with respect to parameters obtained from the results of interpolation calculations have been calculated (Table 3 and Table 4).

Table 3. Cubic Polynomial Parameter Test Results

Name of Parameter	Test size	Critical Values	Comment
a_{01}	3.19	2.04	Significant
a_{10}	10.56	2.04	Significant
a_{20}	4.59	2.04	Significant
a_{11}	|-12.03|	2.04	Significant
a_{02}	|-3.37|	2.04	Significant
a_{30}	7.50	2.04	Significant
a_{21}	|-6.67|	2.04	Significant
a_{12}	8.90	2.04	Significant
a_{03}	2.56	2.04	Significant

Table 4. Quadratic Polynomial Parameter Test Results

Name of Parameter	Test size	Critical Values	Comment
a_{01}	|-6.40|	2.02	Significant
a_{10}	3.18	2.02	Significant
a_{20}	|-1.78|	2.02	Not Significant
a_{11}	|-5.45|	2.02	Significant
a_{02}	4.50	2.02	Significant

Furthermore, by recalculating geoid undulations at the 33 test points with interpolation methods, RMS values are determined. Respectively, for cubic and quadratic polynomial interpolation methods RMS values are calculated ±9.72, ±12.00cm.

6. CONCLUSIONS

In the study, for the chosen study area, it is aimed that the most appropriate method is determined by analyzing outlier measurements in reference points, the validity of modelling parameters and the used methods with the statistical analyses. In the conventional outlier method, when results are examined, It is seen that outliers observations determined in Robust estimation method and Tau test methods are similar. Also, cubic polynomial method has been seen to be appropriate model with 95 % confidence interval at the results of hypothesis test of polynomial parameters computed by using consistent observations. Respectively, for cubic and quadratic polynomial interpolation methods, RMS values are calculated ±9.72, ±12.00cm.

Finally, the results of interpolation carried out the study area are compared; It can be drawn conclusion that cubic interpolation method is better appropriate than the other method.

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BIOGRAPHICAL NOTES

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